PENN ECONOMICS MATH CAMP: PART I

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Contents

1. Binary relations

Sometimes in mathematics we want to work with properties of couples of elements. As economists, we will see this quite often in micro theory.

Definition 1.1. A **binary relation** R over set X is a subset of $X \times X$. We say that an element x is **related** to an element y (denoted by xRy) if (x, y) belongs to this subset.

Example 1.2. "Is greater than or equal to" is a relation: $X = \mathbb{R}$, xRy if and only if $x \geqslant y$.

The following are the most important properties that a relation can have.

Definition 1.3. We say that R is **reflexive** if for any $x \in X, xRx$. We say that R is **symmetric** if for any $x, y \in X$, if xRy then yRx. We say that R is **transitive** if for any $x, y, z \in X$, if xRy and yRz, then xRz.

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*I prepared this set of notes instead of the one that was used in previous years. For the most part, we will cover same topics. I decided to drop the discussion of cuts and construction of the real numbers, as it makes very little practical sense to learn it. Instead, we get a bit deeper into topology, e.g., we discuss nets. I would be extremely grateful for any suggestions or corrections.

Example 1.4. "Is greater than or equal to" is reflexive. Of course, for any $x \in \mathbb{R}, x \geqslant x$. It is also transitive: $x \geqslant y$ and $y \geqslant z$ implies $x \geqslant z$. However, it is not symmetric. $x \geqslant y$ and $y \geqslant x$ cannot hold at the same time for any $x \neq y$.

Example 1.5. "Is strictly greater than" is transitive, but not reflexive and not symmetric.

Exercise 1.6. X is the set of all people. Which of the following relations are reflexive? Symmetric? Transitive?

(1) xRy if x knows y.
(2) xRy if x and y know each other.
(3) xRy if x and y live in the same building.

The following class of relations is a generalization of a very simple relation "is equal to": xRy if and only if $x = y$.

Definition 1.7. A binary relation is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Equivalence relations usually are denoted with $\sim$.

Exercise 1.8. Which of the relations from exercise 1.6 are equivalence relations?

2. Cardinality

It is intuitive that a set of 3 elements is smaller than, say, a set of 7 elements. It is also intuitive that the latter is smaller than $\mathbb{Z}$, which is infinite. But what about $\mathbb{N}$ vs. $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$? Turns out, there also is some way to compare those.

Definition 2.1. A mapping $f : X \to Y$ is an **injection** (or **one-to-one**) if for any $a, b \in X$ if $a \neq b$ then $f(a) \neq f(b)$.

Example 2.2. $f : \mathbb{R} \to \mathbb{R}, x \mapsto \arctan x$ is an injection. $g : \mathbb{R} \to [0, 1], x \mapsto \sin(x)$ is not an injection.

Definition 2.3. A mapping $f : X \to Y$ is a **surjection** (or **onto**) if

$$Y = f(X) = \Big\{ y \in Y | \exists x \in X : f(x) = y \Big\}.$$

Definition 2.4. A mapping is a **bijection** if it is both an injection and a surjection.

Example 2.5. $f : \mathbb{R} \to [0, 1], x \mapsto x^3$ is a bijection.

Exercise 2.6. Show that $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ is a bijection too.

Exercise 2.7. Show that $f : X \to Y$ and $g : Y \to Z$ both are bijections, then $g \circ f : X \to Z$ is a bijection too.

If A is finite and consists of n elements, we usually denote it as $\#A = n$. It is clear that if $\#A = n$ and $\#B = n$, then there is a bijection between them. It is also obvious that if $\#A = n$ and $\#B = m \neq n$, there cannot be a bijection between A and B.

However, we can also think of infinite sets in a similar manner. If there is a bijection between X and Y, then there is a way to pair elements of X with elements of Y such that each element in X has only one pair in Y. Intuitively, it is almost like the two sets "have the same number of elements". We can formalize that now.

Definition 2.8. Two sets A and B are said to have **equal cardinality** if there is a bijection from A to B.

We denote $A \sim B$ or $\#A = \#B$. The former is not a coincidence: equal cardinality is an equivalence relation[1].

Proposition 2.9. *Having the same cardinality is an equivalence relation.*

Proof. Reflexivity is simple and provided by the identity map $id : A \to A, x \mapsto x$. The relation is symmetric as if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection. Finally, if $A \sim B$ and $B \sim C$ with bijections f and g, then $A \sim C$ with the bijection $g \circ f$. □

This is important to build up some intuition about cardinality. In particular, you should not think that it is the same as "physical size". For example, the interval between 0 and 1 has the same cardinality as the interval between -10^{10} and 10^{10}.

Proposition 2.10. *For any real numbers $a < b$, the interval $(a, b) \subset \mathbb{R}$ has the same cardinality as $\mathbb{R}$.*

Proof. First, let's show that $(a, b) \sim (0, 1)$. One of possible bijections is given by

$$f(x) = \frac{x - a}{b - a}.$$

So we also know that $(0, 1) \sim (-\frac{\pi}{2}, \frac{\pi}{2})$. Finally, $x \mapsto \arctan x$ is a bijection between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$. Using the fact that $\sim$ is transitive, we obtain $(a, b) \sim \mathbb{R}$. □

Definition 2.11. A set A is called **denumerable** if $\#A = \#\mathbb{N}$. A set is called **countable** if it is finite or denumerable. A set is called **uncountable** if it is not countable.

A bijection between $\mathbb{N}$ and some set A essentially means that there is a way to enumerate all elements of A. For example, $\mathbb{Z}$ is obviously denumerable (and countable). You just count its elements in the following order: $0, 1, -1, 2, -2, 3, -3, 4 \ldots$ Many important sets are countable, which might seem a bit surprising at first.

Proposition 2.12. $\mathbb{N} \times \mathbb{N}$ *is denumerable.*

Proof. We can enumerate all elements of $\mathbb{N} \times \mathbb{N}$ by going through them in the following fashion: $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots$ If you plot it, you see that we just walk through diagonals on the plane. □

This clearly implies that $A \times B$ is denumerable for any denumerable A and B. By induction, any finite product (e.g., $\mathbb{N}^n$) of denumerable spaces is denumerable.

The following propositions show that there is no infinite set with cardinality smaller than that of $\mathbb{N}$.

Proposition 2.13. *If $A \subset B$ is infinite and B is denumerable, then A is denumerable.*

[1]However, this is not an equivalence relation over a set. That is because "all sets" are not a set. If you are interested, you might want to read about Russell's paradox, which is simple and elegant. The paradox was very confusing for mathematicians for a while. But, as David Hilbert said, "no one shall drive us from the paradise which Cantor has created for us" (Cantor was the creator of set theory). Soon there came Zermelo–Fraenkel set theory and made the rules of the game clear.

Proof. We have a way to enumerate all elements of B. Let us set $f(1)$ to be the element with the smallest number in A, $f(2)$ to be the element with the second smallest number, and so on. It is clear that f is a bijection from $\mathbb{N}$ to A. $\square$

Corollary 2.14. *If $f : A \to B$ is a surjection from denumerable A on infinite B, then B is denumerable.*

Proof. Consider

$$f^{-1}(y) = \Big\{x \in A | f(x) = y\Big\}$$

for every y and choose a unique element in it for each y. Because we have a surjection, this is a correctly defined function. Let us denote it g. By construction, it is an injection of B into A, and by the previous statement $g(B)$ (between which and B there g establishes a bijection) is denumerable. $\square$

Proposition 2.15. *A denumerable union of denumerable sets is denumerable.*

Proof. We have $A = \cup_i A_i$ with A_i being denumerable. Let $f_i : \mathbb{N} \to A_i$ be the corresponding bijections. Then

$$f : \mathbb{N} \times \mathbb{N} \to A, (i, j) \mapsto f_i(j)$$

is a surjection from a denumerable set on an infinite set A. Thus, A is denumerable too. $\square$

Corollary 2.16. $\mathbb{Q}$ *is denumerable. Also,* $\mathbb{N}^n, \mathbb{Z}^n, \mathbb{Q}^n$ *are denumerable for any natural n.*

Proof.

$$f : \mathbb{N} \times \mathbb{N}\backslash\{0\} \to \mathbb{Q}, (i, j) \mapsto \frac{i}{j}$$

is a surjection from a denumerable set on an infinite set, and so $\mathbb{Q}$ is denumerable. $\mathbb{N}^n, \mathbb{Z}^n, \mathbb{Q}^n$ are all finite products of denumerable sets and are denumerable. $\square$

We now have many examples of countable sets. What about uncountable sets? To answer this question, we first should learn how to compare cardinalities.

Definition 2.17. A has **cardinality less than or equal to** the cardinality of B (denoted by $\#A \leqslant \#B$) if there is an injection from A to B.

Definition 2.18. A has **cardinality less than** the cardinality of B (denoted by $\#A < \#B$) if there is an injection from A to B but no bijection between A and B.

Example 2.19. If $\#A = n$ and $\#B = m$, then A has cardinality less than or equal to the cardinality of B if $n \leqslant m$ (so this is consistent to write $\#A \leqslant \#B$). Similarly, A has cardinality less than the cardinality of B if $n < m$ (so this is consistent to write $\#A < \#B$).

Example 2.20. If A is finite, then $\#A < \#\mathbb{N}$.

The following object is very important.

Definition 2.21. The set of all subsets of set A is called the **power set** of set A and is denoted as 2^A.

Proposition 2.22. *If A is finite, then*

$$\#2^A = 2^{\#A}.$$

Proof. For every element of A, we can either take it into a subset or not. So we can encode subsets with vectors of the form $(0,1,0,\ldots)$ of length $\#A$. The number of such vectors is $2^{\#A}$. $\square$

It is clear that for every $n \in \mathbb{N}$, $2^n > n$. But we also can say something about infinite sets.

Theorem 2.23 (Cantor's theorem)**.** *For every set A, $\#2^A > \#A$.*

Proof. First, we have an injection $x \mapsto \{x\}$, which maps A to $\#2^A$. Thus, $\#2^A \geqslant \#A$. We have to show that there is no any bijection from A to $\#2^A$. Suppose there is such a bijection f. Consider

$$B = \Big\{x \in A | x \notin f(x)\Big\}.$$

If f is a bijection, then there exists $y \in A$ such that $f(y) = B$. But this is a contradiction: if $y \in B$, then $y \notin f(y) = B$ by the definition of B. If $y \notin B$, then $y \in f(y) = B$ by the definition of B.

So there is no any bijection, and thus $\#2^A > \#A$. $\square$

In particular, $\#2^{\mathbb{N}} > \#\mathbb{N}$, and $2^{\mathbb{N}}$ is uncountable. Your most important takeaway from this section should be the following fact.

Proposition 2.24. *Cardinality of $\mathbb{R}$ is larger than the cardinality of $\mathbb{N}$ and is the same as the cardinality of $2^{\mathbb{N}}$ (this cardinality is called **continuum**).*

Proof. You can write down any real number in the binary numeral system. Say, $\sqrt{2} = 1.01101010\ldots$, where the sequence of digits can be finite or infinite. We know that $(0,2) \sim \mathbb{R}$, and for numbers from $(0,2)$ the expansion always starts with 0. For every number in $(0,2)$ we have a sequence after the point that consists of 0s and 1s. Just as with finite sets, every sequence of this kind encodes a subset of $\#2^{\mathbb{N}}$. This provides a bijection between $(0,2)$ and $\#2^{\mathbb{N}}$.

This proof is not rigorous enough but provides all the main logic and intuition for why $\#\mathbb{R} = \#2^{\mathbb{N}}$. The problem with the argument above is related to the fact that sometimes two sequences can represent the same number: e.g., in the decimal system $0.99999\ldots = 1$. Strictly speaking, what I described above is not a bijection, and one should make some adjustments to it to make the argument correct. $\square$

The following fact about cardinality also confused mathematicians for some period in history.

Proposition 2.25. *Cardinality of $\mathbb{R}^n$ is the same as cardinality of $\mathbb{R}$ for any n.*

Proof. Again, this will be informal as we will ignore some complications with decimal representation. That said, this completely illustrates the logic. If you have a number $x = 0.a_1a_2a_3\ldots a_na_{n+1}\ldots$, you can map it into an $n-$dimensional vector $(x_1,\ldots,x_n)$ by taking

$$\begin{gathered} x_1 = 0.a_1a_{n+1}a_{2n+1}\ldots, \\ x_2 = 0.a_2a_{n+2}a_{2n+2}\ldots, \\ \ldots \\ x_n = 0.a_na_{2n}a_{3n}\ldots. \end{gathered}$$

This yields a "bijection" between $[0,1]$ and $[0,1]^n$, and thus a bijection between $\mathbb{R}$ and $\mathbb{R}^n$. $\square$

Finally, the following statement is somewhat intuitive but we do not provide a proof here.

Theorem 2.26 (Cantor–Schröder–Bernstein theorem)**.** *If* $\#A \leqslant \#B$ *and* $\#B \leqslant \#A$*, then* $\#A = \#B$*.*

Corollary 2.27. $[0,1], (0,1]$*, a circle on the plane, a sphere in the 3-dimensional space all have the same cardinality as* $\mathbb{R}$*.*

Proof. All these sets belong to some continuum set, and at the same time it is easy to find an injection from $\mathbb{R}$ into them. □

3. Normed spaces

Cardinality basically is the only thing you can say about a set unless it has some additional structure. In the rest of this course, we will consider the structures that are most important from the economics prospective.

An extremely important object are vector spaces. The following properties are intuitive once you have some experience with vector spaces. We list them here but it is not required that you remember all the axioms, let alone their names.

Definition 3.1. A **vector space** (over $\mathbb{R}$) is a set V equipped with two operations: vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$. The operations should satisfy the following properties:

(1) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$ (associativity of $+$).
(2) $u + v = v + u$ for all $u, v \in V$ (commutativity of $+$).
(3) There is $0 \in V$ such that $u + 0 = u$ for all $u \in V$ (identity element for $+$).
(4) For all $u \in V$ there is $-u$ such that $u + (-u) = 0$ (inverse element for $+$).
(5) $\lambda(\mu u) = (\lambda\mu)u$ for all $u \in V$, $\lambda, \mu \in \mathbb{R}$ (compatibility of multiplications).
(6) $1u = u$ for all $u \in V$ (identity element of scalar multiplication).
(7) $(\lambda+\mu)u = \lambda u+\mu u$ for all $u \in V$, $\lambda, \mu \in \mathbb{R}$ (distributivity of scalar addition).
(8) $\lambda(u+v) = \lambda u+\lambda v$ for all $u, v \in V$, $\lambda \in \mathbb{R}$ (distributivity of vector addition).

Example 3.2. $\mathbb{R}^n$ is a vector space.

Example 3.3. The space of infinite sequences $(x_1, x_2, \dots)$ is a vector space.

Example 3.4. $C([a, b])$, the set of all continuous functions over closed interval $[a, b]$, is a vector space.

You will spend more time on vector spaces in the second part of the course. Here, we are interested in their metric and topological structure. The following definition is a generalization of vector's length.

Definition 3.5. Suppose V is a vector space. A **norm** is a function $\|\cdot\| : V \to \mathbb{R}_+$ such that

(1) $\|v\| = 0$ if and only if $v = 0$.
(2) $\|\alpha v\| = |\alpha| \, \|v\|$ for any $v \in V, \alpha \in \mathbb{R}$.
(3) $\|v + u\| \leqslant \|v\| + \|u\|$for any $v, u \in V$.

Example 3.6. The **Euclidean norm** of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is given by

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Example 3.7. The **sup norm** of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is given by

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

Exercise 3.8. Show that it is indeed a norm.

Example 3.9. The **Manhattan norm** of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is given by

$$\|x\|_1 = |x_1| + \ldots + |x_n|.$$

Exercise 3.10. Show that it is indeed a norm.

Example 3.11. One can define a norm on $C([a, b])$ (which is also called **sup norm**) as

$$\|f\| = \max_{x \in [a,b]} \Big| f(x) \Big|.$$

The norm exists because any continuous function over a closed interval attains a maximum.

Exercise 3.12. Show that it is indeed a norm.

4. Metric spaces

The following concept is a generalization of distance.

Definition 4.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}_+$ such that

(1) $d(x, y) = d(y, x)$ for any $x, y \in X$ (symmetry)
(2) $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles)
(3) $d(x, z) \leqslant d(x, y) + d(y, z)$ for any $x, y, z \in X$ (triangle inequality).

Definition 4.2. A **metric space** is a set X equipped with a metric d over it.

Example 4.3. Let $(V, \|x\|)$ be a normed vector space. Then this norm induces a metric over V given by $d(x, y) = \|x - y\|$.

Example 4.4. In particular, for $\mathbb{R}^n$ with the standard norm this becomes the standard Euclidian distance. For $\mathbb{R}$, it is just $d(x, y) = |x - y|$.

Example 4.5. Similarly, $C([a, b])$ becomes a metric space with the metric given by

$$d(f, g) = \max_{x \in [a,b]} \Big| f(x) - g(x) \Big|.$$

The following example is less important for economists but provides a good illustration of how powerful is the concept. Also, it is a metric not generated by a norm.

Example 4.6. The Levenshtein distance between two strings of characters x and y is the minimal number of deletions, insertions, or substitutions required to transform x into y.

You should try to utilize your geometric intuition while working with metric spaces. Elements of a metric space are often called points for a similar reason.

4.1. **Sequences in metric spaces.** Sequence is just a set of points indexed by natural numbers.

Definition 4.7. A **Cauchy sequence** is a sequence of points $a_i \in M$ with the property that $\forall \epsilon > 0 \; \exists \; N$ such that $d(x_n, x_m) < \epsilon$ if $n, m > N$.

The points of a Cauchy sequence get closer and closer to each other. This happens, for example, if a sequence converges to some point.

Definition 4.8. A sequence (a_i) **converges** to a if

$$\forall \epsilon > 0 \; \exists \; N : \forall i > N \; d(a_i, a) < \epsilon.$$

We use the notation $a_i \to a$ and call a a **limit** of the sequence.

Exercise 4.9. The limit of a converging sequence is unique.

Proposition 4.10. *Every converging sequence in a metric space is a Cauchy sequence.*

Proof. Just use the definition of convergence and choose N such that $d(a_i, a) < \epsilon/2$ for $i > N$. Then,

$$d(a_n, a_m) \leqslant d(a_n, a) + d(a, a_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n, m > N$. □

Sometimes, every Cauchy sequence converges.

Theorem 4.11. *If (a_i) is a Cauchy sequence in $\mathbb{R}$, then it converges.*

Proof. The proof relies a lot on the fundamental properties or real numbers, which are listed in the appendix. The proof is in several steps.

(1) Let us choose N such that $|a_n - a_m| < \epsilon/3$ for $n, m > N$. Then,

$$|a_{N+1} - a_n| < \epsilon/3 \implies a_{N+1} - \epsilon/3 < a_n < a_{N+1} + \epsilon/3$$

for all $n > N$. From that we obtain that the sequence is bounded.

(2) Because it is bounded from below, there exists $\underline{a}_n = \inf\{a_i\}_{i \geqslant n}$. Because it is bounded from above, there exists $\bar{a}_n = \sup\{a_i\}_{i \geqslant n}$. We have

$$\underline{a}_n \leqslant \underline{a}_{n+1} \leqslant \bar{a}_{n+1} \leqslant \bar{a}_n$$

since $\{a_i\}_{i \geqslant n+1} \subset \{a_i\}_{i \geqslant n}$. In other words, $[\underline{a}_n, \bar{a}_n] \supset [\underline{a}_{n+1}, \bar{a}_{n+1}] \supset \dots$, and nested closed intervals have a non-empty intersection[2]. Let us choose a point a in this intersection. It will be unique and it will be the limit.

(3) We have $a, a_n \in [\underline{a}_n, \bar{a}_n]$, and thus $|a - a_n| < \bar{a}_n - \underline{a}_n$. From step 1, if N is such that for $n > N$

$$a_{N+1} - \epsilon/3 < a_n < a_{N+1} + \epsilon/3,$$

then $\underline{a}_n \geqslant a_{N+1} - \epsilon/3$ and $\bar{a}_n \leqslant a_{N+1} + \epsilon/3$, and thus

$$|a - a_n| \leqslant \bar{a}_n - \underline{a}_n \leqslant \epsilon/3 + \epsilon/3 < \epsilon.$$

□

That inspires the following definition.

[2] See the appendix. You also can convince yourself by drawing pictures.

Definition 4.12. A metric space M is called **complete** if any Cauchy sequence converges to a point in M.

Example 4.13. If you restrict the Euclidean metric from $\mathbb{R}$ on $M = (0, 1)$, you obtain a metric space that is not complete. The sequence

$$a_i = \frac{1}{i}$$

would converge to 0 in $\mathbb{R}$ but 0 is not an element of M.

Example 4.14. If you restrict the Euclidean metric from $\mathbb{R}$ on $\mathbb{Q}$, you obtain a metric space that is not complete. Every irrational number can be approximated by rational numbers with any degree of precision, and thus one can construct a sequence in $\mathbb{Q}$ that converges, say, to $\sqrt{2}$. This would be a Cauchy sequence (because it converges in $\mathbb{R}$), but it does not converge to a point in the metric space.

Another important example of a complete space is $\mathbb{R}^n$. To show that it is complete, first you should notice that elements of $\mathbb{R}^n$ converge if they converge coordinate-wise.

Exercise 4.15. Let $a_i = (a_i^1, \ldots, a_i^n)$ be a sequence in $\mathbb{R}^n$. Then,

$$a_i \to a = (a^1, \ldots, a^n)$$

if and only if $a_i^k \to a^k$ for all $k = 1, \ldots, n$.

Proposition 4.16. $\mathbb{R}^n$ *with the Euclidean metric is complete.*

Proof. If (a_i) is a Cauchy sequence, the (a_i^k) also is a Cauchy sequence for any k:

$$|a_n^k - a_m^k| = \sqrt{(a_n^k - a_m^k)^2} \leqslant \sqrt{\sum_{i=1}^{n} (a_n^i - a_m^i)^2} = d(a_n, a_m) < \epsilon.$$

Thus, a_i^k converges to some a^k, and $a_i \to a = (a^1, \ldots, a^n)$. □

The last example we consider will be the following.

Proposition 4.17. $C([a, b])$ *is a complete metric space.*

Proof. Suppose (f_n) is a Cauchy sequence of functions. Then, for all x

$$|f_n(x) - f_m(x)| \leqslant d(f_n, f_m),$$

and this shows that $f_n(x)$ is a Cauchy sequence of real numbers. We set $f(x)$ to be the limit of this sequence for each x. Now, let's show that $f_n \to f$. If N is such that for $n, m > N$ we have $d(f_n, f_m) < \epsilon$, then for all $x \in [a, b]$

$$|f(x) - f_n(x)| = \lim_{i \to \infty} |f_i(x) - f_n(x)| \leqslant \epsilon,$$

and thus if the maximum exists, it is bounded by ϵ:

$$\max_{x \in [a,b]} \Big| f(x) - f_n(x) \Big| \leqslant \epsilon$$

for $n > N$. The last step is to show that f also is a continuous function. Suppose we want $|f(x_0) - f(x)| < \delta$. We can choose N such that $|f(x) - f_N(x)| < \delta/3$ for all $x \in [a, b]$. Also, $f_N \in C([a, b])$ and thus we can choose ϵ such that $|f_N(x_0) - f_N(x)| < \delta/3$ for $|x - x_0| < \epsilon$. Then, for $|x - x_0| < \epsilon$ we have

$$|f(x_0) - f(x)| \leqslant |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| <$$

$$\delta/3 + \delta/3 + \delta/3 = \delta.$$

□

Exercise 4.18. Show that the set of differentiable functions over $[a, b]$ is not complete.

Exercise 4.19. One can define another metric over $C([a, b])$ as

$$d(f, g) = \int_a^b |f(x) - g(x)| dx.$$

The metric space obtained here is not complete.

4.2. **Contractions.** An extremely important application of metrics is due to contractions. Contractions work well in complete spaces. Consider a metric space (M, d).

Definition 4.20. A **contraction** is a mapping $A : M \to M$ for which there exists $\lambda \in [0, 1)$ such that for all $x, y \in M$

$$d(A(x), A(y)) \leqslant \lambda d(x, y).$$

Intuitively, points get closer to each other after A is applied. Importantly, there is some λ that provides an upper bound for this.

Example 4.21. $A : \mathbb{R} \to \mathbb{R}, A(x) = 0.5x + 1$ is a contraction for $\lambda = 0.5$.

Notice that A has a fixed point: $A(2) = 2$. This is not a coincidence.

Theorem 4.22 (Banach fixed-point theorem)**.** *Let A be a contraction in a complete metric space. Then A has a unique fixed point $x^* \in M$ such that $A(x^*) = x^*$. The sequence $x, A(x), A^2(x), \ldots$ converges to x^* for any x.*

Proof. Suppose $n = m + k \geqslant m$. For any point x,

$$d(A^n(x), A^m(x)) \leqslant \lambda d(A^{n-1}(x), A^{m-1}(x)) \leqslant \ldots \leqslant \lambda^m d(A^k(x), x) \leqslant$$

$$\lambda^m \Big[d(A^k(x), A^{k-1}(x)) + d(A^{k-1}(x), A^{k-2}(x)) + \ldots + d(A(x), x) \Big] \leqslant$$

$$\lambda^m \Big[\lambda^k + \lambda^{k-1} + \ldots + 1 \Big] d(A(x), x) = \lambda^m \frac{1 - \lambda^k}{1 - \lambda} < \frac{\lambda^m}{1 - \lambda}.$$

Because $\lambda^m \to 0$, we obtained that $x, A(x), A^2(x), \ldots$ is a Cauchy sequence. The space is complete, so it has some limit x^*. This will be the fixed point. Obviously, the sequence $A^{n+1}x$ should converge to the same point as $A^n x$. But also

$$d(A^{n+1}(x), A(x^*)) < \lambda d(A^n(x), x^*)$$

shows that it converges to $A(x^*)$. Thus, $A(x^*) = x^*$.

Finally, the fixed point is unique. Suppose there is another one, x^{**}. Then,

$$d(x^*, x^{**}) = d(A(x^*), A(x^{**})) \leqslant \lambda d(x^*, x^{**}),$$

which is possible only for $d(x^*, x^{**}) = 0$, which implies $x^* = x^{**}$. □

The theorem is extremely useful because it shows that a fixed point exists. Suppose you are interested in existence of some object. You can prove it in three steps. First, you show it lives in a complete space (usually, this is related to completeness of $C([a, b])$). Second, you show that this object is a fixed point of some mapping.

Third, you show that this mapping is a contraction. Below is a list of most important examples. These are not even sketches of proofs, but they are supposed to show you how important Banach fixed point theorem is.

Example 4.23 (Cauchy problem)**.** Consider ordinary differential equation

$$\frac{df}{dt} = v(f,t).$$

Here, M would be a set of functions with some properties. $A : M \to M$ is defined as

$$Ag(t,x) = \int_0^\tau v(\tau, x + g(t,x))d\tau$$

and can be shown to be a contraction. It has a fixed point g^*, and then $f = x + g^*(t,x)$ is a solution to the differential equation.

Example 4.24 (Inverse function theorem)**.** Loosely speaking, if f has a continuous non-zero derivative at x, f^{-1} exists around $f(x)$. You can show that by defining a proper contraction.

Example 4.25 (Implicit function theorem)**.** Given some $F(x,y)$, you want to show existence of a function $y(x)$ such that $F(x,y(x)) = 0$ for all x. This again is done by defining a proper contraction.

Example 4.26 (Bellman equation)**.** If you discount your future at rate β and behave rationally, your value at point x satisfies

$$V(x) = \max_a \Big[u(a,x) + \beta V(x')\Big].$$

This is a functional equation on V, and existence of the solution follows from the fact that

$$A : f \mapsto \max_a \Big[u(a,x) + \beta f(x')\Big]$$

is a contraction under some conditions.

4.3. **Topology of metric spaces.** It is very simple to generalize the notion of continuity to metric spaces.

Definition 4.27. A function $f : M \to N$ between metric spaces (M, d_M) and (N, d_N) is **continuous at** $x \in M$ if for any $\delta > 0$ there is $\epsilon > 0$ such that for all $y \in M$ with $d_M(x,y) < \epsilon$ it holds that $d_N(f(x), f(y)) < \delta$. A function is **continuous** if it is continuous for all $x \in M$.

Example 4.28. For $M, N = \mathbb{R}$ this is just the standard definition of continuity from basic calculus.

Exercise 4.29. For a vector space V with a metric given by a norm, $\|\cdot\| : V \to \mathbb{R}$ is continuous.

Exercise 4.30. Show that any contraction is continuous.

For continuity and convergence, what matters is not the values of d between particular points, but the fact that d becomes infinitely small. This means that you actually can find another "language" for working with continuity that is not going to depend on the metric. We now start to learn this language.

Think of a ball without its boundary. The following definition generalizes this geometric concept for an arbitrary metric space.

Definition 4.31. An **open ball** with center x and radius $\epsilon > 0$ (also called $\epsilon-$**neighborhood** of x) is the subset defined as

$$B_\epsilon(x) = \Big\{ y \in X | d(x, y) < \epsilon \Big\}.$$

Consider the following system of subsets of X:

$$\tau_d = \Big\{ A \subset X | \forall a \in A \ \exists \epsilon > 0 : B_\epsilon(x) \subset A \Big\}.$$

These are such subsets A that every point of A is contained in A with some $\epsilon-$neighborhood. We call the sets from τ_d **open sets**.

Example 4.32. Suppose $X = \mathbb{R}$ with the standard Euclidian metric. Then, $(0, 1) \subset \mathbb{R}$ is open, while $[0, 1] \subset \mathbb{R}$ is not.

τ_d has very important properties that are going to inspire an extremely important definition. Let's take a look at these properties.

Proposition 4.33. *The following is true about open sets:*

(1) *$X, \emptyset$ are open.*
(2) *Every union of open sets is open. That is, for any indexing set A and any $U_\alpha \in \tau_d$*
$$\Big(\bigcup_{\alpha \in A} U_\alpha \Big) \in \tau_d.$$
(3) *Every finite intersection of open sets is open. That is, for any $U_1, \dots, U_n \in \tau_d$*
$$\Big(\bigcap_{i=1}^{n} U_i \Big) \in \tau_d.$$

Proof. This properties are based on the definition of a $\epsilon-$neighborhood.

(1) There is nothing to check for $\emptyset$. Also, $B_\epsilon(x) \subset X$ by definition for any $x \in X$.
(2) If $x \in U = \Big(\cup_{\alpha \in A} U_\alpha \Big)$, then $x \in U_\alpha$ for some open U_α. We just take the neighborhood of x in this U_α. It also is contained in the union as $U_\alpha \subset U$.
(3) Suppose $x \in \Big(\cap_{i=1}^{n} U_i \Big)$. Consider neighborhoods of x in all the n sets: $B_{\epsilon_1}(x) \subset U_1, \dots, B_{\epsilon_n}(x) \subset U_n$. Let us take $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Because n is finite, ϵ is correctly defined and positive. Also, $B_\epsilon(x) \subset B_{\epsilon_i}(x)$ for $i = 1, \dots, n$. Thus, $B_\epsilon(x) \subset \Big(\cap_{i=1}^{n} U_i \Big)$.

□

We can rewrite the definition of continuity in terms of open sets. Intuitively, a function is continuous if for all x we can find an open neighborhood within which the image of f is close to $f(x)$.

Proposition 4.34. *$f : X \to Y$ is continuous at $x \in X$ if and only if for any open $V, f(x) \in V$ there exists an open neighborhood $U, x \in U$ such that $f(U) \subset V$.*

Proof. Suppose $f : X \to Y$ is continuous at x and we have $V, f(x) \in V$. By definition, there is an open ball with center $f(x)$ in V. Let it have radius δ. Now just take ϵ from the definition of continuity. Then $U = B_\epsilon(x)$ is an open neighborhood of x and

$$f(B_\epsilon(x)) \subset B_\delta(f(x)) \subset V.$$

Now suppose the condition is satisfied and we have some δ. Since $B_\delta(f(x))$ is open, there is U such that

$$f(U) \subset B_\delta(f(x)).$$

There is an open ball $B_\epsilon(x) \subset U$, and this ϵ is such that for all $y \in X, d(y,x) < \epsilon$ we have $d_N(f(x), f(y)) < \delta$. □

This obviously implies

Proposition 4.35. *$f : X \to Y$ is continuous if and only if for any x and any open $V, f(x) \in V$ there exists an open neighborhood $U, x \in U$ such that $f(U) \subset V$.*

5. Topology

The properties we considered above turn out to be relevant for many objects, not just open subsets of metric spaces. At the same time, they capture something very important about continuity and proximity. This is why we will get more abstract now. Consider some set X.

Definition 5.1. A **topology** τ on X is a system of subsets of X such that

(1) $X, \emptyset \in \tau$.

(2) For any indexing set A and any $U_\alpha \in \tau$,

$$\left(\bigcup_{\alpha \in A} U_\alpha\right) \in \tau.$$

(3) For any $U_1, \ldots, U_n \in \tau$

$$\left(\bigcap_{i=1}^{n} U_i\right) \in \tau.$$

Open subsets of a metric space X are a topology. Similarly, once we have a topology, we call sets from τ **open**.

Definition 5.2. A **topological space** is a pair (X, τ), where τ is a topology on X.

Another important class of sets in a topological space is the following.

Definition 5.3. A set $A \subset X$ is **closed** if $X \setminus A$ is open.

The following is way to rephrase the definition of topology in terms of closed sets.

Exercise 5.4. The following statements are true about topological spaces:

(1) $X, \emptyset$ are closed.

(2) For any indexing set A and any closed B_α,

$$\bigcap_{\alpha \in A} B_\alpha$$

is closed.

(3) For any closed $B_1, \ldots, B_n$,

$$\bigcup_{i=1}^{n} B_i$$

is closed.

It is important that you have some intuition for what is closed and what is open.

Example 5.5. Suppose $X = \mathbb{R}$ with the standard Euclidian metric. Then, $[0,1] \subset \mathbb{R}$ is closed. $(0,1] \subset \mathbb{R}$ or $[0,1) \subset \mathbb{R}$ are neither open nor closed.

However, we can define topological spaces without using metrics.

Example 5.6. Consider any set X. Then,

$$\tau = \Big\{X, \emptyset\Big\} \subset 2^X$$

is a topology, which is called the **trivial topology**.

Example 5.7. Consider any set X. Then, $\tau = 2^X$ is a topology, which is called the **discrete topology**.

Exercise 5.8. Consider $X = \mathbb{N}$. Let open sets be subsets B such that $\mathbb{N} \setminus B$ is finite. This is a topology on X.

Exercise 5.9. Suppose (X, τ) is a topological space. Consider $Y = X \cup \{a\}$ and subsets of type $U \cup \{a\}$, for $U \in \tau$. Show that all such subsets and $\emptyset$ are a topology on Y.

In calculus, when worked with continuity we usually worked with open balls. But in general balls are not better than open sets, while the former are present in any topological space.

Definition 5.10. Any open set U such that $x \in U$ is an **open neighborhood** of x.

The definition of a topology is very general, and topological spaces satisfying it can be very different. To get some results, we need to assume additional properties. The following property says that every two points can be "separated".

Definition 5.11. A topological space is called **Hausdorff** if for any $x, y \in X$ there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.

Example 5.12. The standard topology on $\mathbb{R}$ is Hausdorff. The discrete topology on every set is Hausdorff. The trivial topology is not Hausdorff, just as the topology from example 5.9.

Exercise 5.13. Show that points are closed sets in any Hausdorff space.

5.1. **Continuous functions.** For metric spaces, we got equivalent conditions of continuity. We will use them as the definition of continuity for topological spaces.

Definition 5.14. A mapping $f : X \to Y$ is **continuous at** $x \in X$ if and only if for any open $V, f(x) \in V$ there exists an open neighborhood $U, x \in U$ such that $f(U) \subset V$. A mapping $f : X \to Y$ is **continuous** if it is continuous at any $x \in X$.

Proposition 5.15. *A mapping $X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for any open $V \subset Y$.*

Proof. If $f^{-1}(V)$ is open in X for any open V, then for all $V \ni x$ we trivially have $f(f^{-1}(V)) = V \subset V$, and thus f is continuous.

If f is continuous and V is open, then for any $x \in f^{-1}(V)$ there is an open neighborhood U_x of x such that $f(U_x) \subset V$. Thus, $U_x \subset f^{-1}(V)$. This implies

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

and is open as a union of open sets. □

Example 5.16. $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is continuous. For $b > a > 0$ we have $f^{-1}\Big((a, b)\Big) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$, which is an open set.

Corollary 5.17. *A mapping $X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for any closed $C \subset Y$.*

Proof. If f is continuous, then the inverse images of closed sets

$$f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$$

are closed. If the inverse images of closed sets are closed, then the inverse images of open sets

$$f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$$

are open. □

These two results have both theoretical and applied importance. For example, many sets we use in economics are solutions of systems of equations and inequalities. We now can easily tell when they are closed (or open).

Example 5.18. The set

$$\Big\{(x, y) \in \mathbb{R}^2 | x^2 + \frac{y^2}{4} \leqslant 1\Big\}$$

is closed as the inverse image of a closed set $(-\infty, 1]$ under continuous function $f(x, y) = x^2 + y^2/4$. The set

$$\Big\{(x, y) \in \mathbb{R}^2 | x^2 + \frac{y^2}{4} < 1\Big\}$$

is open as the inverse image of an open set $(-\infty, 1)$.

The following statement you should remember from calculus. However, it was a bit of an exercise. Using the language of topology, we can prove it in one line.

Proposition 5.19. *If $f : X \to Y$ and $g : Y \to Z$ are continuous functions between topological spaces, then $g \circ f : X \to Z$ is continuous too.*

Proof. For any open $V \subset Z$,

$$(g \circ f)^{-1}(U) = f^{-1}\Big(g^{-1}(U)\Big)$$

is open as $g^{-1}(U)$ is open and f is continuous. □

Corollary 5.20. *For any continuous $f, g : X \to \mathbb{R}$, fg, λf, and $\lambda f + \mu g$ are continuous for all constants $\lambda, \mu \in \mathbb{R}$.*

5.2. **Comparing topologies.** Suppose we have two topologies. The following definition is fascinating but we will not be able to dedicate a lot of attention to it.

Definition 5.21. A mapping $f : X \to Y$ is a **homeomorphism** if it is continuous and f^{-1} is continuous too. Then, X and Y are said to be **homeomorphic**.

Exercise 5.22. Show that $(0, 1)$ is homeomorphic to $\mathbb{R}$.

Think about X and Y that are homeomorphic. Suppose that some topological statement[3] about X is true. Because f makes a one-to-one correspondence between open sets of X and Y, this statement automatically becomes true for Y. Essentially, you just renamed all points of X into points of Y, and you can do that because the topological structure is preserved. From the topological point of view, X and Y are indistinguishable[4].

The famous example is how a doughnut can be deformed into a coffee cup[5]. Those are homeomorphic, and yes, they have the same topological properties.

The previous was an example when two different sets "had the same topology". But it also can be that the same set has different topologies.

Definition 5.23. Suppose τ_1 and τ_2 are both topologies on X. If

$$\tau_1 \subset \tau_2,$$

then τ_1 is **weaker** than τ_2, and τ_2 is **stronger** than τ_1.

Example 5.24. The discrete topology is stronger than any other topology. The trivial topology is weaker than any other topology.

Exercise 5.25. Show that τ_1 is stronger than τ_2 if and only if $id : x \mapsto x$ is a continuous function between (X, τ_1) and (X, τ_2).

5.3. **More topological spaces.** Though you already might be disappointed by how many topological spaces there are, we will now learn to construct new topological spaces from topological spaces.

Definition 5.26. Suppose Z is a subset of a topological space X with topology τ. The **induced topology** on Z is given by

$$\tau_Z = \Big\{U \cap Z | U \in \tau\Big\}.$$

Then, (Z, τ_Z) is called a **topological subspace** of (X, τ)

Exercise 5.27. Show that τ_Z is a topology.

Exercise 5.28. Show that $Z \to X, z \mapsto z$ is a continuous function from (Z, τ_Z) to (X, τ).

For some cases, there is a natural way to explicitly describe the open sets in a topology. But you also can think about its "smaller pieces". These pieces will convey all the information about the structure of the topology. It is simpler sometimes to define a topology or prove results about it in terms of these smaller pieces.

Definition 5.29. A collection $\mathcal{B}$ of open sets is a **base** of topology τ if any open set from τ is a union of sets from $\mathcal{B}$.

Exercise 5.30. $\mathcal{B}$ is a base if and only if for any $U \in \tau$ and any $x \in U$ there exists $V \in \mathcal{B}$ such that $x \in V$.

Exercise 5.31. Show that all open intervals are a base for the topology on $\mathbb{R}$.

[3]That is, a statement that can be formulated in terms of open sets. For example, think about the definition of compactness. It is is formulated in terms of open sets. This is also the case for most of the other definitions we discuss in this section.

[4]This is some profound idea in mathematics. If you are interested, you can read about isomorphisms and category theory.

[5]See Wikipedia.

If $\mathcal{B}$ is a base, then X is open and thus is a union of sets from $\mathcal{B}$. Also, for any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is open and thus is a union of sets from $\mathcal{B}$. Now consider an abstract collection of subsets with these properties.

Proposition 5.32. *Suppose that $\mathcal{B} \subset 2^X$ is such that*

(1) X is a union of elements from $\mathcal{B}$.
(2) For any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is a union of sets from $\mathcal{B}$.

Then, $\mathcal{B}$ is a base of some topology on X.

For a topological space, we might have many bases. And if we have a collection with the two properties of bases, we might use it to define a topological space.

Definition 5.33. Consider two topological spaces, (X, τ_X) and (Y, τ_Y). A **product topology** $\tau_X \times \tau_Y$ on $X \times Y$ is the topology with the base

$$\mathcal{B} = \Big\{ U_X \times U_Y | U_X \in \tau_X, U_Y \in \tau_Y \Big\}.$$

Exercise 5.34. Show that $\mathcal{B}$ is indeed a base.

Exercise 5.35. Show that there might be other open sets in $\tau_X \times \tau_Y$ than the sets from $\mathcal{B}$.

Finally, let us think about $\mathbb{R}^n$, the most important sets we know. What would be a topology there? We have the Euclidian norm, which gives us the Euclidian distance. It generates some topology. We have the *sup*$-$norm on $\mathbb{R}^n$. It also generates a topology. We could multiply the Euclidian norm by 2 and get another metric, and thus another topology. Finally, $\mathbb{R}^n$ has the product topology from $\mathbb{R}^{n-1}$ and $\mathbb{R}$. Which topology is the one we prefer? Fortunately, they all will be the same. We could show that using bases. Also, an important result (which we will not prove) is that any two norms on a finite-dimensional vector space generate the same topology.

Theorem 5.36. *Any two norms on $\mathbb{R}^n$ generate the same topology.*

5.4. **Sets in a topological space.** Not all sets in a topological space are open or closed. But we can turn them into.

Definition 5.37. The **closure** of set E is defined as

$$\bar{E} = \bigcap_{E \subset C, C \text{ is closed}} C.$$

This is the smallest closed set that contains E. The following is the largest open set that is contained in E.

Definition 5.38. The **interior** of set E is defined as

$$Int(E) = \bigcup_{U \subset E, U \in \tau} U.$$

Definition 5.39. The **boundary** of set E is defined as

$$\partial E = \bar{E} \setminus Int(E).$$

Exercise 5.40. Why those are correct definitions?

Exercise 5.41. Show that $\bar{E}$ is the set of all points $x \in E$ such that any open neighborhood of x has a non-empty intersection with E.

Exercise 5.42. Show that $Int(E)$ is the set of points that belong to E with some open neighborhood.

Exercise 5.43. What is the interior, closure, and boundary of
$(0,1), \{0,1\}, [0,1], (0,1] \subset \mathbb{R}$?

Some points can be approximated by points in a set.

Definition 5.44. A **limit point** of set E is a point x such that any open neighborhood of x contains some point in E other than x.

A limit point is always in the closure, but not all points in the closure are limit points (think of a singleton in $\mathbb{R}$, for example, which is closed but has no limit points). The term is a bit problematic. For example, not all limits of sequences are limit points.

Proposition 5.45. *A point x in a Hausdorff space is a limit point of E if and only if any open $U \ni x$ contains infinitely many points of E.*

Proof. It is obvious that the latter implies the former in any space. Now assume that x is in a Hausdorff space. If $U \ni x$ contains finitely many points $x_1, \ldots, x_n$ of E, we can consider

$$U \bigcap X \setminus \{x_1, \ldots, x_n\} \ni x,$$

which is also open but has no points of E. Thus, x is not a limit point. □

Example 5.46. Any point of $[0,1]$ is a limit point.

Exercise 5.47. Show that a set is closed if and only if it contains all its limit points.

6. Compacts

You know a lot of compacts already, but we will be a bit abstract about the concept at first. Later, you will learn a very nice characterization of compacts in metric spaces.

Definition 6.1. An **open cover** of set $B \subset X$ is a collection of open sets U_α such that

$$B \subset \bigcup_\alpha U_\alpha.$$

A **subcover** consists of some but not necessarily all sets from a cover.

Definition 6.2. A subset $B \subset X$ is a **compact** if for any open cover of B, there is a finite subcover.

This definition might seem very abstract. However, the concept of a compact actually will be used in every part of your first-year economics sequence – in macro, in micro, and in econometrics. Let's start with examples of sets that are not compact.

Example 6.3. $\mathbb{R}$ is not compact. $\mathbb{N} \subset \mathbb{R}$ is not compact.

An extremely important example of a compact set are closed intervals.

Example 6.4. For all real numbers $a < b$, the closed interval $[a,b]$ is compact.

Why it is compact? First, this is a fact from your calculus class (it probably was stated in a more simple language there). However, this also will be a corollary of a more general result we will prove later. For now, just hold this example in mind.

Proposition 6.5. *For every topological space, if $B \subset X$ is finite then it is compact.*

Proof. Just take an open neighborhood from the cover for any element of the set. □

Thus, compact sets generalize finite sets. We might hope that some good properties of finite sets will be preserved by compact sets. This is indeed the case. A simple illustration is the following. A finite union of finite sets is finite. For compact sets, we have the same.

Proposition 6.6. *A finite union of compact sets is compact. That is, if $B_1, \dots, B_n \subset X$ are compact, then $B = \cup_{i=1}^{n} B_i$ is compact.*

Proof. Just take the finite subcovers for $B_1, \dots, B_n$. This is a finite subcover of B. □

We now will prove some statements about compacts. All of them will be (relatively) simple implications of the finite subcover property.

Proposition 6.7. *If $K \subset X$ is a compact subspace of a Hausdorff space X, then K is closed.*

Proof. We should prove that $X \setminus K$ is open. Consider any $x \in X \setminus K$. We should find an open neighborhood for x such that it has empty intersection with K. For every $y \in K$ we have neighborhoods $x \in U_y, y \in V_y$ such that $U_y \cap V_y = \emptyset$. V_y will be an open cover of K, and by assumption we can choose a subcover $V_{y_1}, \dots, V_{y_n}$. Then, $U = U_{x_1} \cap \dots \cap U_{x_n}$ is an open neighborhood of x. It has empty intersection with $V = V_{y_1} \cup \dots V_{y_n}$, and thus with $K \supset V$ too. □

Example 6.8. The closed interval $[a, b] \subset \mathbb{R}$ is compact and $\mathbb{R}$ is Hausdorff. The closed interval is closed.

Now let's think about the following. For example, $[a, b] \subset \mathbb{R}$ is compact as a subset of $\mathbb{R}$. However, $[a, b]$ also is a topological space itself with the topology induced from $\mathbb{R}$. Is it also compact there? The answer is positive. The special property of compactness is that it is invariant. This is not the case for other topological definitions: say, (a, b) is closed in topological subspace (a, b) but it is not a closed set in $\mathbb{R}$.

Proposition 6.9. *If $K \subset X$ is compact, then K is a compact in the induced topology on K.*

Proof. Suppose V_α are a cover of K in (K, τ_K). By definition of the induced topology, for every α we know that $V_\alpha = U_\alpha \cap K$ for some open $U_\alpha \subset X$. Clearly, U_α are a cover of K. Take a finite subcover $U_1, \dots, U_n$. Then, $V_1, \dots, V_n$ are a finite subcover of K in (K, τ_K). □

We will also use that to prove other facts about compacts, assuming for simplicity but WLOG that K is the whole space.

Proposition 6.10. *If $L \subset K$ is closed and K is compact, then L also is compact.*

Proof. If we have an open cover of L, we can add $K \setminus L$ to it and obtain an open cover of K. Take a finite subcover and exclude $K \setminus L$. It will be a finite subcover of L. □

Proposition 6.11. *If $K_X \subset X, K_Y \subset Y$ are compact, then $K_X \times K_Y$ is compact in $X \times Y$.*

Proof. Assume we have an open cover of $K_X \times K_Y$. Take from it some open set $V_{(x,y)} \ni (x, y)$ for each $(x, y) \in K_X \times K_Y$. Because of the structure of product topology, there should be an open neighborhood $U_{X,(x,y)} \times U_{Y,(x,y)} \ni (x, y)$.

Now let's fix y and consider $U_{X,(x,y)}$ for all x. They cover K_X, and thus there is a finite subcover $U_{X,(x_1,y)}, \ldots, U_{X,(x_{n(y)},y)}$ of K_X. Consider

$$U_{Y,y} = U_{Y,(x_1(y),y)} \cap \ldots \cap U_{Y,(x_{n(y)}(y),y)} \ni y.$$

This is an open set, and we have such a set for each $y \in K_Y$. Again we have an open cover and can take a finite subcover $U_{Y,y_1}, \ldots, U_{Y,y_m}$. Sets

$$V_{(x_1(y_1),y_1)}, \ldots, V_{(x_{n(y_1)}(y_1),y_1)}, \ldots, V_{(x_{n(y_m)}(y_m),y_m)}$$

are a finite subcover of $K_X \times K_Y$. You might wonder why. Suppose we have a point (x, y). We just unwind the sequence of steps we did. There is some $U_{Y,y_i} \ni y$. Then for each y, in particular, for $y = y_i$,

$$x \in U_{X,(x_j(y),y)} = U_{X,(x_j(y_i),y_i)}$$

for some x_j. Finally,

$$y \in U_{Y,y_i} = U_{Y,(x_1(y_i),y_i)} \cap \ldots \cap U_{Y,(x_{n(y_i)}(y_i),y_i)} \subset U_{Y,(x_j(y_i),y_i)},$$

which implies

$$(x, y) \in U_{X,(x_j,y_i)} \times U_{Y,(x_j,y_i)} \subset V_{(x_j,y_i)},$$

which is in the subcover we chose. □

Corollary 6.12. *Boxes $[a_1, b_1] \times \ldots \times [a_n, b_n]$ are compacts in $\mathbb{R}^n$.*

Previously, we considered finite Cartesian products of topological spaces.

Definition 6.13. If X_α is a system of topological spaces, then the **product topology** on their Cartesian product X is the topology with the base consisting of

$$\prod_\alpha U_\alpha \subset \prod_\alpha X_\alpha = X,$$

where U_α is open in X_α and only a finite number of U_α are different from X_α.

The following result is very important (at least for mathematicians) but we cannot prove it in this course.

Theorem 6.14 (Tychonoff's theorem)**.** *Any product of compact spaces is compact in the product topology.*

The following property again shows that compactness is special.

Proposition 6.15. *If $K \subset X$ is compact and $f : X \to Y$ is a continuous mapping, then $f(K)$ is compact.*

Proof. Suppose $f(K)$ is covered by open V_α. Consider $U_\alpha = f^{-1}(V_\alpha)$, which is an open cover of K. Since K is compact, take a finite subcover $U_1, \dots, U_n$. Then, $V_1, \dots, V_n$ are a finite cover of K:

$$f(K) \subset f(U_1) \cup \dots \cup f(U_n) = V_1 \cup \dots \cup V_n.$$

□

Finally, the following way to reformulate compactness is useful.

Definition 6.16. A collection of sets has **finite intersection property** if any finite intersection of some of its members is not empty.

Theorem 6.17. *X is compact if and only if any collection of closed sets with finite intersection property has nonempty intersection.*

Proof. First, let us notice the following. Suppose C_α is a system of closed sets and $U_\alpha = X \setminus C_\alpha$ is the system of its open compliments. Then C_α have empty intersection exactly when U_α are a cover, as

$$\bigcup_\alpha U_\alpha = X \setminus \Big(\bigcap_\alpha C_\alpha\Big).$$

And a finite subcover exists exactly when the finite intersection property is not satisfied, as

$$X = \bigcup_{i=1}^n U_i = X \setminus \Big(\bigcap_{i=1}^n C_i\Big).$$

The rest is almost tautological. Let X be compact and C_α be a collection of closed sets with finite intersection property. Then we consider $U_\alpha = X \setminus C_\alpha$ and find that it cannot be a cover as otherwise we would have a finite subcover, which would be a contradiction with the finite intersection property.

Now, suppose any collection of closed sets with the finite intersection property has nonempty intersection. Suppose we have an open cover of X with U_α. Again, we switch to closed sets and consider $C_\alpha = X \setminus U_\alpha$. Because the finite intersection property cannot be satisfied (otherwise there would be a nonempty intersection, and we would not have a cover) for some $C_1, \dots, C_n$ we should have empty intersection and then $U_1, \dots, U_n$ is a finite subcover among U_α. □

Corollary 6.18. *Suppose $K_1 \supset K_2 \supset \dots$ are all compact and nonempty in a Hausdorff space. Then,*

$$\bigcap_{i=1}^\infty K_i$$

is not empty.

Proof. Because the space is Hausdorff, these are closed sets. The system satisfies the finite intersection property as $K_{i_1} \cap \dots \cap K_{i_n} = K_i$ for $i = \max\{i_1, \dots, i_n\}$. □

Exercise 6.19. Provide an example of $B_1 \supset B_2 \supset \dots$ such that they are in a Hausdorff space (say, $\mathbb{R}$), not compact, and $\bigcap_{i=1}^\infty B_i$ is empty.

Example 6.20. A sequence of nested closed intervals has a nonempty intersection (we used that when we proved that $\mathbb{R}$ is complete).

Proposition 6.21. *Any infinite set in a compact topological space has a limit point.*

Proof. We always can choose a denumerable subset $\{x_n\}_{n=1}^{\infty}$, and it is enough to show that we can find a limit point for it. Consider sets $C_n = \{x_n, x_{n+1}, \ldots\}$. If this set has no limit points, then it is closed (any point in the complement has a neighborhood that is disjoint with C_n). Then $C_1 \supset C_2 \supset C_3 \supset \ldots$ satisfy the infinite intersection property but have empty intersection, which contradicts compactness. □

For practical purposes, to understand what compacts are we will need to restrict ourselves to metric spaces.

7. Topology of metric spaces

The following definition generalizes sequences to topological spaces.

Definition 7.1. A sequence of points (x_n) converges to x if for any open $U \ni x$ there exists N such that $x_n \in U$ for all $n > N$.

Exercise 7.2. Show that for metric spaces this is the same as the previous definition of convergence.

Metric spaces allow to use sequences to answer topological questions. Below, we always assume the topology induced by the metric. The following sequence of results is quite intuitive. First, we characterize closed sets.

Proposition 7.3. *For a metric space M and any subset E, $\bar{E}$ is the set of limits of converging sequences in E.*

Proof. $\bar{E}$ is the set of all points x such that every open neighborhood of x has a nonempty intersection with E.

First, if x is a limit of a sequence in E then this property should be satisfied. Otherwise assume $U \ni x$ and $U \cap E = \emptyset$. Then, there should be N such that $x_n \in U$ for all $n > N$, but this is not possible for $x_n \in E$.

Second, assume x satisfies the property. Consider open balls $B_{1/n}(x)$. Each of them should have some intersection with E, and choosing $x_n \in B_{1/n}(x) \cap E$ we obtain a sequence in E that converges to x. □

Corollary 7.4. *In a metric space, E is closed if and only if it contains all its sequential limits.*

Corollary 7.5. *If $A \subset M$ is closed and M is a complete metric space, then A is complete.*

We also can characterize continuity.

Proposition 7.6. *A function $f : M \to N$ is continuous at x if for any sequence $x_n \to x$ we have $f(x_n) \to f(x)$.*

Proof. If f is continuous, then for any open $V \ni f(x)$ we have some open set $U \ni x$ such that $f(U) \subset V$. As $x_n \to x$, there is N after which x_n is in U. Then $f(x_n) \in V$ for $n > N$. We get that $f(x_n) \to f(x)$.

Now suppose $f(x_n) \to f(x)$ for any $x_n \to x$ but f is not continuous at x. This means there is U such that for any $\epsilon_n = 1/n$ there is $x_n \in B_{\epsilon_n}(x)$ with $f(x_n) \notin U$. But then $f(x_n)$ does not converge to $f(x)$, which is a contradiction. □

7.1. **Compacts in metric spaces.** We also can characterize compacts. But that is a bit harder. Because balls are at the core of the definition of the metric topology, we will use them a lot in this section.

Definition 7.7. A subset $A \subset M$ is **totally bounded** if for any $\epsilon > 0$ there is a finite set $\{x_1, \dots, x_n\} \subset M$ such that

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

(the set is $\epsilon-$**dense**).

Example 7.8. $\mathbb{N}^2$ is $\epsilon-$dense in $\mathbb{R}^2$ for any $\epsilon > \frac{1}{\sqrt{2}}$.

Example 7.9. In $\mathbb{R}^n$, totally bounded sets are exactly **bounded** sets, that is, sets A such that for some N they belong to a large box:

$$A \subset [-N, N]^n \subset \mathbb{R}^n.$$

Proof. Every subset of a totally bounded set is totally bounded. So it is enough to show that for boxes. But boxes you clearly can split into very small boxes and take vertices of those. So boxes are totally bounded. And if you have a set that is totally bounded, then clearly the distance from 0 to all points in this set is smaller then

$$\max_i d(0, x_i) + \epsilon,$$

and this set is in some large box. □

There is a definition that is very related to compactness.

Definition 7.10. Subset $A \subset X$ is **sequentially compact** if any sequence in A has a subsequence that converges to an element of A.

In metric spaces, this will turn out to be the same as compactness. But in an abstract topological space, sequential compacts might not be compact and compacts might not be sequentially compact. There are other attempts to define sets that have behavior similar to that of compacts. But we will not go there. For now, consider the following.

Lemma 7.11. *If U_α is an open cover of a sequentially compact metric space, then for some $\epsilon > 0$ we have that for all x there is α such that*

$$B_\epsilon(x) \subset U_\alpha.$$

*This ϵ is called a **Lebesgue number** of the cover.*

Proof. Suppose there is no Lebesgue number. Then for any n we have some x_n such that $B_{1/n}(x_n) \not\subset U_\alpha$ for all α. Suppose there is a subsequence with limit x. Since we have an open cover, for some α we have $x \in U_\alpha$. Then, there is some ball $B_{\frac{1}{m}}(x) \subset U_\alpha$. But if x is the limit of a subsequence, it also means that we always can find some $x_n \in B_{\frac{1}{2m}}(x)$ for arbitrary large n. But if

$$\frac{1}{n} + \frac{1}{2m} < \frac{1}{m},$$

then $B_{1/n}(x_n) \subset B_{1/m}(x) \subset U_\alpha$ and we have a contradiction with the definition of x_n.

□

We need this lemma for the following.

Theorem 7.12. *Suppose M is a metric space. Then the following statements are equivalent:*

(1) M is compact.
(2) M is complete and totally bounded.
(3) Every sequence in M has a convergent subsequence (M is sequentially compact).

Proof. We will show that (1) implies (2), (2) implies (3), and (3) implies (1). Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) Suppose M is compact. First,

$$M = \bigcup_{x \in M} B_\epsilon(x),$$

and we can choose a finite subcover for any ϵ. Thus, M is totally bounded. Now let's show that M is complete. Consider a Cauchy sequence (x_n). If it has a finite set of values, then clearly it is constant after some point (just take the minimum of the distances between the points and apply the Cauchy property). It has a limit, of course. If (x_n) is infinite, then it has a limit point x. This point is going to be the limit. For any $\epsilon > 0$ we can choose N such that $n, m > N$ implies $d(x_n, x_m) < \epsilon/2$. And because $B_{\epsilon/2}(x)$ has infinitely many points of the sequence, there is x_M in it with $M > N$. Then,

$$d(x_n, x) \leqslant d(x_n, x_M) + d(x_M, x) \leqslant \epsilon/2 + \epsilon/2 = \epsilon,$$

and $x_n \to x$.

(2) Suppose M is complete and totally bounded. Consider some sequence (a_n). Consider closed balls $B_1^c(y_1), \ldots, B_1^c(y_n)$ that cover M. One of them should have infinitely many values of the sequence as the indexing set is infinite. Take this ball $B_1^c(y_i)$ and the subsequence of values in it. We denote the center of the ball as $x_1 = y_i$ and the subsequence as (a_n^1). This ball, as a subset of a totally bounded set, is totally bounded and covered by some closed balls $B_{1/2}^c(y_1), \ldots, B_{1/2}^c(y_m)$. Again take the one that has infinitely many values and the corresponding subsequence (a_n^2) of (a_n^1). Now consider closed balls

$$A_k = B^c_{2*1/2^k}(x_k) \supset B^c_{1/2^k}(x_k).$$

They are closed and they are nested[6] as for any $x \in A_{k+1}$,

$$d(x, x_k) \leqslant d(x, x_{k+1}) + d(x_{k+1}, x_k) \leqslant 2 * 1/2^{k+1} + 1/2^k = 2 * 1/2^k.$$

We also have $x_m \in A_k$ for $m \geqslant k$. Thus, this is a Cauchy sequence. It has some limit a. Because the sequence belongs to closed sets A_k for $n \geqslant k$, the limit belongs to each A_k too. Thus, the point belongs to

$$\bigcap_{i=1}^{n} A_i,$$

[6] We consider them because we want a sequence of nested sets.

and we easily can construct a subsequence of (a_n) that converges to a. Just take some element of (a_n^1), then take an element with larger index from (a_n^2), and so on. Each kth element is in A_k, and we converge to a.

(3) Now suppose every sequence in M has a convergent subsequence. From lemma 7.11 we know there is a Lebesgue number ϵ. If we find $\{x_1, \dots, x_n\}$ such that
$$M \subset \bigcup_{i=1}^{n} B_\epsilon(x_i),$$
then we just take an α for each x_i and get a finite subcover. Assume we cannot find such $\{x_1, \dots, x_n\}$. Let us start with some point x_1. By the assumption, there is x_2 in M farther then ϵ from x_1. Take it. Then
$$M \not\subset \bigcup_{i=1}^{2} B_\epsilon(x_i),$$
and there is a point x_3 with $d(x_i, x_3) \geqslant \epsilon$ for $i = 1, 2$. Continuing this way, we find a sequence such that $d(x_n, x_m) > \epsilon$ for any n, m. This property is true for any subsequence of (x_i), but a convergent subsequence should be a Cauchy sequence. We get a contradiction.

□

Corollary 7.13. *Closed intervals $[a, b] \subset \mathbb{R}$ are compact.*

Corollary 7.14 (Heine-Borel Theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Now, given this result, we obtain a general version of the Weierstrass extreme value theorem. From the applied perspective, this is the most important result in the course.

Theorem 7.15. *If $f : K \to \mathbb{R}$ is continuous and K is compact, then f attains a maximum and a minimum.*

Proof. We know that $f(K)$ is a compact in $\mathbb{R}$. Thus, it is closed and bounded. Since it is bounded, it has supremum. Since it is closed, it contains the supremum. Thus, the supremum is in the image of f and is a maximum. Same for infinium. □

Exercise 7.16. Is $C([a, b])$ compact? Is it totally bounded?

If we change "for any" and "there exists" in the definition of continuity, we obtain a more restrictive property.

Definition 7.17. A function $f : M \to N$ between metric spaces (M, d_M) and (N, d_N) is **uniformly continuous** if for any $\delta > 0$ there is $\epsilon > 0$ such that for all $x, y \in M$ with $d_M(x, y) < \epsilon$ it holds that $d_N(f(x), f(y)) < \delta$.

Theorem 7.18. *If $f : M \to N$ is continuous and M is compact, then f is uniformly continuous.*

Proof. Suppose it is not. Then we have some $\delta > 0$ and $x_n, y_n \in M$ for any n such that
$$d_M(x_n, y_n) < \frac{1}{n},\ d_N(f(x_n), f(y_n)) > \delta.$$

Since we deal with a compact, (x_n) has converging subsequence $x_{n_k} \to x$. Clearly, $y_{n_k} \to x$ too. But then

$$\delta < d_N(f(x_{n_k}), f(y_{n_k})) \leqslant d_N(f(x_{n_k}), f(x)) + d_N(f(x), f(y_{n_k}))$$

cannot hold for a continuous f. □

Exercise 7.19. Prove the same result in a more conceptual way (without sequences). Use that $d_N \circ (f \times f) : M \times M \to \mathbb{R}$ is continuous and $M \times M$ is compact.

8. Connected sets

The following concept sounds abstract but also has a strong geometric intuition.

Definition 8.1. A topological space (X, τ) is **connected** if the only sets that are both open and closed in it are X and $\emptyset$.

Definition 8.2. A set $A \subset X$ is **connected** if (A, τ_A) is connected.

Example 8.3. $\mathbb{R} \setminus \{c\}$ is not connected for any $c \in \mathbb{R}$.

What are the connected sets in $\mathbb{R}$? Those are intervals, possibly infinite.

Proposition 8.4. *All connected sets in* $\mathbb{R}$ *are either* $[a, b], (a, b], (a, b), [a, b)$ *with* $a, b \in \mathbb{R}$*, or* $(-\infty, a], (-\infty, a), [a, \infty), (a, \infty)$ *with* $a \in \mathbb{R}$*, or* $\mathbb{R}$.

Proof. If you think about this for a while you would agree that this is the same as to say that $A \subset \mathbb{R}$ is connected if and only $\forall x, y \in A$, $[x, y] \subset A$.

(1) First, suppose $E \subset \mathbb{R}$ is connected and $x, y \in E$, but there is a point z between x and y that is not in A. Then $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$ are nonempty and open in (E, τ_E). Thus, $E = A \cup B$ cannot be connected.

(2) Now, suppose all intermediate points belong to E. Let us show that E is connected. Suppose $E = A \sqcup B$[7], where A and B are open (and closed) and nonempty. Take $a \in A, b \in B$. WLOG, we can assume $a < b$. We also have $[a, b] = \tilde{A} \sqcup \tilde{B}$, where $\tilde{A} = [a, b] \cap A$ and $\tilde{B} = [a, b] \cap B$ are open (and closed) in $[a, b]$ and nonempty (one has a, the other has b). Consider

$$l = \sup \tilde{A}.$$

It exists because we deal with a bounded nonempty set. Because $\tilde{A}$ is closed we have $l \in \tilde{A}$, and also $l \geqslant a$. Now consider

$$u = \inf \tilde{B} \cap [l, b].$$

Again it exists because we deal with a bounded nonempty set. Again, because $\tilde{B}$ is closed and thus $\tilde{B} \cap [l, b]$ is closed too, we have $u \in \tilde{B}$, and also $l \leqslant u \leqslant b$. Because $\tilde{A} \cap \tilde{B} = \emptyset$, it should be that $l < u$. But then $(l, u) \cap \tilde{A} = \emptyset$, $(l, u) \cap \tilde{B} = \emptyset$, and thus $(l, u) \cap [a, b] = \emptyset$, which is a contradiction.

□

Intuitively, continuous functions should not "tear apart" connected sets. This is indeed the case.

[7] We use $\sqcup$ for disjoint unions.

Proposition 8.5. *If $f : X \to Y$ is continuous and $E \subset X$ is connected, then $f(E)$ is connected.*

Proof. Suppose $f(E) = A \cup B$, with $A, B \neq \emptyset$ being open in $(f(E), \tau_{f(E)})$. This means that there are open $\tilde{A}, \tilde{B}$ in Y such that $A = f(E) \cap \tilde{A}$, $B = f(E) \cap \tilde{B}$. Then $f^{-1}(\tilde{A})$ and $f^{-1}(\tilde{B})$ are open in X and each of them has some elements from E. But then $f^{-1}(\tilde{A}) \cap E$ and $f^{-1}(\tilde{B}) \cap E$ are both not empty and open in (E, τ_E). Which is a contradiction if E is connected. □

This and proposition 8.4 yield the following result.

Theorem 8.6 (Intermediate value theorem)**.** *If $f : X \to \mathbb{R}$ is continuous and X is connected, then for any $a, b \in X$, f takes all values between $f(a)$ and $f(b)$.*

Proof. Since $f(X)$ is connected and $f(a), f(b) \in f(X)$, all values between $f(a)$ and $f(b)$ are in $f(X)$. □

This fact is basic calculus for $X = \mathbb{R}$, but now we can say that it is also true for much more general class of sets.

The following captures well the intuition about connectedness. Union of connected sets with a common point is connected.

Exercise 8.7. If

$$X = \bigcup_\alpha X_\alpha,$$

$$\bigcap_\alpha X_\alpha \neq \emptyset$$

with each X_α being connected, then X is connected.

The following definition will be very important for the other parts of the course.

Definition 8.8. A subset A of vector space V is **convex** if for any points x, y it includes the **line segment**

$$\Big\{\alpha x + (1-\alpha)y | \alpha \in [0,1]\Big\}$$

between x and y.

Convexity is very important for optimization, but it requires some additional structure over the set. The following definition is very geometric and in some sense generalizes convexity for an abstract topological space.

Definition 8.9. A set E in a topological space is **path-connected** if for every two points x, y there is a continuous function $f : [a, b] \to E$ such that $f(a) = x, f(b) = y$.

Exercise 8.10. Show that being path-connected is an equivalence relation.

Proposition 8.11. *If X is path-connected, then it is connected.*

Proof. Suppose E is path-connected but not connected. Then, $E = A \sqcup B$. Consider two points $x \in A$, $y \in B$ and a path $f : [a, b] \to E$ between them. We have $f^{-1}(A),^{-1}(B)$ being both open and disjoint cover of $[a, b]$, which contradicts $[a, b]$ being connected. □

Proposition 8.12. *All convex sets in a vector space are connected.*

Proof. The line segment between two points of a convex set provides a path between them. Thus, convex sets are path-connected. □

Finally, let's consider the less abstract case of $\mathbb{R}^n$. The following statement might speak to your geometric intuition.

Proposition 8.13. *If $U \subset \mathbb{R}^n$ is open and connected, then it is path-connected.*

Proof. Consider two points $x, y \in U$ and suppose there is no path between them. All points z of U for which there is a path from x to z are an open set. This is because if you have a path to z and $B_\epsilon(z) \subset U$, then you can construct a path to all points in $B_\epsilon(z) \subset U$ (think about the picture here).

Now consider the set of points for which there is no path from x. It is nonempty as it contains y. It is also open. For each z in it, $B_\epsilon(z) \subset U$ for some ϵ, and then no point in $B_\epsilon(z) \subset U$ can be connected to x with a path. Otherwise we could construct a path from x to z.

Thus, we get a disjoint cover of U with two open sets. This is not possible if U is connected. □

Exercise 8.14. Use connectedness to show that $[0, 1]$ is not homeomorphic to the unit circle

$$S^1 = \Big\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\Big\}.$$

9. Nets

Metric spaces essentially allow us to deal only with sequences. But in general, sequences are not enough to answer questions about some basic topological properties. For example, in a metric space a function is continuous if applied to every converging sequence it preserves convergence. This is not the case for an abstract topological space. First, let us see how the things break. We will need the following fancy definition.

Definition 9.1. The **cocountable topology** on X is the topology in which the set is closed if and only if it is countable or coincides with X.

Exercise 9.2. Use the finite intersection property to show that $\mathbb{R}$ with the cocountable topology is not compact.

Clearly, any constant sequence converges. Of course, in general the opposite is not true. But for the cocountable topology it is the case.

Proposition 9.3. *If (x_n) converges in the cocountable topology, (x_n) is constant after some N.*

Proof. Suppose $x_n \to x$. Let us consider $U = X \setminus \{x_n | x_n \neq x\}$. It is an open neighborhood of x, so for some N we should have $x_n \in U$ for all $n > N$. But then it means that $x_n = x$ for all $n > N$ by the definition of U. □

This topology provides two bad examples at the same time.

Example 9.4. Any set in $\mathbb{R}$ with the cocountable topology contains the limit of any converging sequence. This is because converging sequences in a set end up being just constants, and that constants belong to the set. But not any set is closed, e.g., $\mathbb{R}_+$ is not closed.

Example 9.5. Consider $f : \mathbb{R} \to \mathbb{R}$, where the domain has the cocountable topology and the codomain has the standard topology. For every converging sequence $x_i \to x$ we have $f(x_i) \to f(x)$, because $f(x_i)$ are eventually constant. But f is not continuous. $(0, 1)$ is open in the standard topology, but

$$f^{-1}((0, 1)) = (0, 1)$$

is not open as $\mathbb{R} \setminus (0, 1)$ is not countable.

So sequences do not work as a tool in more general settings. At the same time, you had a chance to see that sometimes it was very convenient to use sequences (say, to prove that the direct product of two metric compacts is compact). To keep some of this convenience, we will need to substitute sequences with something bigger.

If you think about sequences, what we used all the time is the ability to choose indexes larger than some particular index.

Definition 9.6. A **direction** is a binary relation $\succeq$ over set I that is reflexive, transitive and such that for any x, y there is $z \succeq x, y$ (this z is called **upper bound**). Then, I becomes a **directed set**.

Example 9.7. The standard order $\geqslant$ on $\mathbb{R}$ or $\mathbb{N}$ is a direction.

The following example is the most important for the discussion.

Example 9.8. If x is a point in a topological space, then

$$\mathcal{N}_x = \Big\{U \in \tau | x \in U\Big\}$$

is a directed set if we set $U \succeq V$ when $U \subset V$.

Definition 9.9. A **net** is a function $I \mapsto X$, where I is a directed set.

Sequences are nets because $\mathbb{N}$ is a directed set. But nets might be more complex. We also can generalize the notions of convergence and a limit.

Definition 9.10. A net x_α converges to x if for any open $V \ni x$ there is $\beta \in A$ such that $x_\alpha \in V$ for all $\alpha \succeq \beta$.

Example 9.11. Consider some point $x \in X$ and for any open $U \ni x$ choose $x_U \in X$. Then, $(x_U)_{U \in \mathcal{N}_x}$ is a net that converges to x.

Now we can use nets to do the job that sequences fail to do. The following statements should resemble to you the facts you know about metric spaces.

Proposition 9.12. *If $x \in \bar{E} \subset X$ for some set E if and only if there is a net x_α in E such that $x_\alpha \to x$.*

Proof. First, suppose for some net $x_\alpha \to x$ for x_α in E. If $x \notin \bar{E}$ then there is a neighborhood U of x with $E \cap U = \emptyset$. For some β we should have $x_\alpha \in U$ for all $\alpha \succeq \beta$, and by the definition of a direction such an x_α indeed exists. But this is a contradiction with $x_\alpha \in E$ for all α.

Second, suppose $x \in \bar{E}$. This means that no neighborhood U of x has empty intersection with E, and there is some $x_U \in U \cap E$. All neighborhoods of x are a directed set and thus we obtain a net x_U with $x_U \in E$. For any $V \succeq U$ we have $x_V \in V \subset U$ and thus $x_U \to x$. □

Proposition 9.13. *A function $f : X \to Y$ is continuous at x if for any net $x_\alpha \to x$ we have $f(x_\alpha) \to f(x)$.*

Proof. First, suppose f is continuous and we are given $V \ni f(x)$. Consider $f^{-1}(V)$ and β such that $x_\alpha \in f^{-1}(V)$ for all $\alpha \succeq \beta$. Then, $f(\alpha) \in V$ for all $\alpha \succeq \beta$. Thus, $f(x_\alpha) \to f(x)$.

Now, suppose the condition is satisfied but f is not continuous, and thus $f^{-1}(V)$ is not open for some open V. This means that $X \setminus f^{-1}(V)$ is not closed, and there is a point x in its closure that is not in the set. This means there is a net x_α in $X \setminus f^{-1}(V)$ that converges to $x \in f^{-1}(V)$. Then $f(x_\alpha) \to f(x)$ for $f(x_\alpha) \in Y \setminus V$ and $f(x) \in V$. This is not possible because $Y \setminus V$ is closed. □

All metric spaces are Hausdorff, but in general converging sequences and nets do not have a unique limit. We can use nets to characterize Hausdorff spaces.

Proposition 9.14. *Topological space X is Hausdorff if and only if when $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.*

Proof. First, suppose the space is Hausdorff but there are two limits x and y for a net x_α. Then, we can find $U \ni x, V \ni y$ with $U \cap V = \emptyset$. There is $\beta_U \in I$ such that $x_\alpha \in U$ for all $\alpha \succeq \beta_U$. And there is $\beta_V \in I$ such that $x_\alpha \in V$ for all $\alpha \succeq \beta_V$. But also there is $\beta \succeq \beta_U, \beta_V$. It should be $x_\beta \in U, x_\beta \in V$, but that is not possible.

Now, suppose every converging net has unique limit but the space is not Hausdorff. There should be x, y such that for all neighborhoods $U \ni x, V \ni y$ their intersection $U \cap V$ is nonempty. Choose some $x_{(U,V)} \in U \cap V$. Define a net using the direction

$$(U, V) \succeq (U', V') \Leftrightarrow U \subset U', V \subset V'.$$

This net converges to both x and y, but this is not possible by the assumption. □

Subsequences proved to be important, and we will need their counterpart for nets. Think of σ as of some kind of choice function that maps $\mathbb{N}$ into the chosen indexes of the sequence.

Definition 9.15. We have a **subnet** $(y_j)_{j\in J}$ of net $(x_i)_{i\in I}$ if there is a function $\sigma : J \to I$ such that

1. For all $j \in J$, $x_{\sigma(j)} = y_j$.
2. For all $i \in I$ there is $j \in J$ such that $\sigma(j) \succeq i$.
3. For all $j_1 \succeq j_2$, $\sigma(j_1) \succeq \sigma(j_2)$.

Example 9.16. If the original net was some sequence (x_n), then this definition allows for more subnets than just subsequences. E.g., $(x_1, x_1, x_2, x_2, x_2, x_3, \ldots)$ is a subnet of (x_n) but not a subsequence. Here, $\sigma(1) = 1, \sigma(2) = 1, \sigma(3) = 2$, and so on.

Not all compacts are sequential compacts, but there is a related characterization.

Theorem 9.17. *Topological space X is compact if and only if for every net x_α in it there is a converging subnet.*

Proof. (1) First, suppose the space is compact. Suppose we are given a net. Consider all points "above" i: $A_i = \{x_j | j \succeq x_i\}$. Because of the definition of a net, they satisfy the finite intersection property. Thus, their closures $\bar{A}_i$ satisfy it too. Now we can take

$$y \in \bigcap_{i \in I} \bar{A}_i.$$

Because of the properties of closures, it means that for any open $U \ni y$ we have $U \cap A_i \neq \emptyset$, which actually means that for any i there is $j \succeq i$ with $x_j \in U$. We are ready to provide a subnet now. The indexing set is

$$J = \Big\{(i, U) \in J \times \mathcal{N}_y | x_i \in U\Big\},$$

and the direction on it is given by

$$(i, U) \succeq (i', U') \Leftrightarrow i \succeq i', U \subset U'.$$

An upper bound exists because of our choice of y. Consider

$$\sigma : (i, U) \to i.$$

The subnet we obtain converges to y, as for any $(i, U) \succeq (i', V)$ we have $x_{\sigma((i,U))} = x_i \in U \subset V$.

(2) Now, suppose we always have a converging subnet. Suppose X is not compact, and so for some C_α with the finite intersection property we have

$$\bigcap_{\alpha \in A} C_\alpha = \emptyset.$$

Let I be all finite subsets of A. The direction on A is provided by inclusion (just as with $\mathcal{N}_x$). Because of the final intersection property, we can choose

$$x_i \in \bigcap_{\alpha \in i} C_\alpha$$

for each i. We get a net, and this net should have a converging subnet (for which we will use the same notation J, σ) by the assumption. Let y be a limit of this subnet, and because the intersection is empty, we have $y \notin C_\alpha$ for some α. Because C_α is closed, for some open $U \ni y$ we have $U \cap C_\alpha = \emptyset$. This is going to be problematic. For some j for all $j' \succeq j$ in J we should have $x_{\sigma(j')} \in U$. Also, because we deal with a subnet, for some j'' we have $\sigma(j'') \succeq \{\alpha\} \in I$. Thus, for j such that $j \succeq j', j''$ we have $x_{\sigma(j)} \in U$ and at the same time

$$x_{\sigma(j)} \in \bigcap_{\beta \in \sigma(j)} C_\beta \subset \bigcap_{\beta \in \{\alpha\}} C_\beta = C_\alpha,$$

which is a contradiction with $U \cap C_\alpha = \emptyset$.

□

Exercise 9.18. Use this characterization of compactness to find a much easier proof of the fact that the Cartesian product of two compacts is compact.

10. Correspondences

Suppose you play a game against a single opponent. She chooses a strategy, which is the same as to choose a point in the strategy space. Now you choose your best response. Sometimes it is a single strategy, but often there would be a set of strategies that yield the best payoff you can obtain. Thus, you actually have a "function" that might have multiple values. This is the object we are going to consider now.

Definition 10.1. A **correspondence** ϕ between two sets X and Y maps each point $x \in X$ into a subset $\phi(x) \subset Y$ (in other words, this is a function from X to 2^Y).

Notation 10.2. We will denote correspondences with $\phi : X \rightrightarrows Y$.

We will be interested in correspondences between topological spaces. To do something with the topological structure, we need to find out what continuous correspondences are. But for set-valued functions the definition of continuity splits into two definitions. The first one says that the sets cannot become larger in a discontinuous way.

Definition 10.3. A correspondence $\phi : X \rightrightarrows Y$ is **upper hemicontinuous** at x if for any open $V \supset \phi(x)$ there exists an open neighborhood $U \ni x$ such that for all $y \in U$ it holds that $\phi(y) \subset V$.

The second one says that the sets cannot become smaller in a discontinuous way.

Definition 10.4. A correspondence $\phi : X \rightrightarrows Y$ is **lower hemicontinuous** at x if for any open V such that $V \cap \phi(x) \neq \emptyset$ there exists an open neighborhood $U \ni x$ such that for all $y \in U$ it holds that $V \cap \phi(y) \neq \emptyset$.

The examples below illustrate these definitions. Note that it is kind of easy to confuse what is "discontinuously larger" and what is "discontinuously smaller".

Example 10.5. Consider $\phi : [0,1] \rightrightarrows [0,1]$, with $\phi(x) = \{0\}$ for $x < 1$ and $\phi(1) = [0,1]$. This correspondence is upper hemicontinuous but is not lower hemicontinuous at $x = 1$.

Example 10.6. Consider $\phi : [0,1] \rightrightarrows [0,1]$, with $\phi(x) = [0,1]$ for $x < 1$ and $\phi(1) = \{0\}$. This correspondence is lower hemicontinuous but is not upper hemicontinuous at $x = 1$.

Notation 10.7. It saves a lot of time to write **uhc** instead of upper hemicontinuous and **lhc** instead of lower hemicontinuous.

If you like to check continuity through openness of inverse images of open sets, the following two statements are counterparts of the continuity properties for functions.

Exercise 10.8. A correspondence $\phi : X \rightrightarrows Y$ is upper hemicontinuous if and only if its **upper inverse**, defined as

$$\phi^u(V) = \Big\{x \in X | \phi(x) \subset V\Big\},$$

is open for any open $V \subset Y$.

Exercise 10.9. A correspondence $\phi : X \rightrightarrows Y$ is lower hemicontinuous if and only if its **lower inverse**, defined as

$$\phi^l(V) = \Big\{x \in X | \phi(x) \cap V \neq \emptyset\Big\},$$

is open for any open $V \subset Y$.

Of course, every function f also defines a correspondence $\phi(x) = \{f(x)\}$. Then, upper hemicontinuity of ϕ would be the same as lower hemicontinuity of ϕ and the same as continuity of f. But in general we should use these more complicated definitions.

Definition 10.10. A correspondence $\phi : X \rightrightarrows Y$ is **continuous** if it is both upper hemicontinuous and lower hemicontinuous.

Example 10.11. Consider $\phi : [0, 1] \rightrightarrows [0, 1]$, with $\phi(x) = [0, x]$. This correspondence is continuous.

Some things about functions keep being true for correspondences.

Proposition 10.12. *Suppose $\phi : X \rightrightarrows Y$ is an upper hemicontinuous correspondence with compact values, and $K \subset X$ is a compact. The image of K, defined as*

$$\phi(K) = \bigcup_{x \in K} \phi(x),$$

is compact.

Proof. Suppose V_α is an open cover of $\phi(K)$. For each $x \in K$, $\phi(x)$ is compact and covered by V_α, and we can choose a finite subcover. Consider union V_x of these subcovers. Now, $\phi^u(V_x)$ is open because ϕ is uhc. All $\phi^u(V_x)$ are a cover of K, and we can choose a finite subcover because K is compact. Let it be $\phi^u(V_{x_1}), \dots, \phi^u(V_{x_n})$.

We have

$$\phi(K) \subset \phi\Big(\phi^u(V_{x_1}) \cup \ldots \cup \phi^u(V_{x_n})\Big) =$$

$$\phi\Big(\phi^u(V_{x_1})\Big) \cup \ldots \cup \phi\Big(\phi^u(V_{x_n})\Big) \subset V_{x_1} \cup \ldots \cup V_{x_n}.$$

Thus, we can the finite subcovers for $x_1, \dots, x_n$ altogether are a finite subcover of $\phi(K)$. We used the fact that $\phi\Big(\phi^u(A)\Big) \subset A$ for any A when ϕ is uhc. □

When you plot a function from $\mathbb{R}$ to $\mathbb{R}$, you use a graph on the plane $\mathbb{R}^2$. This inspires to the following generalization.

Definition 10.13. The **graph** of a correspondence $\phi : X \rightrightarrows Y$ is the subset of $X \times Y$ defined as

$$Gr_\phi = \Big\{(x, y) \in X \times Y | y \in \phi(x)\Big\}.$$

It is convenient to work with graphs of correspondences, and thus the following result is important.

Theorem 10.14 (Closed Graph Theorem)**.** *Suppose $\phi : X \rightrightarrows Y$ is a correspondence and Y is compact and Hausdorff. Then ϕ has a closed graph if and only if ϕ is upper hemicontinuous and has closed values.*

Proof. (1) First, suppose ϕ is upper hemicontinuous and has closed values. We will show that any point $(x, y) \notin Gr_\phi$ has an open neighborhood that is disjoint with Gr_ϕ. By definition, $y \notin \phi(x)$. Since $\phi(x)$ is a closed subset of compact Y, it is compact. In proposition 6.7 we actually shown that if a point does not belong to a compact, we can find disjoint open neighborhoods for it and the compact. Thus, there are open $V \ni y$, $W \supset \phi(x)$ such that $V \cap W = \emptyset$. Consider $U = \phi^u(W)$. It is open and $U \times W$ is open in $X \times Y$. Since $x \in U, y \in V$, it is an open neighborhood of (x, y). At the same time, if $(a, b) \in U \times W$ then $\phi(a) \in W$, $b \in V$, and thus $b \notin \phi(a) \implies (a, b) \notin Gr_\phi$.

(2) Now suppose that ϕ has a closed graph. It is trivial that $\phi(x)$ is closed for any x. We will show that this only is possible if ϕ is upper hemicontinuous. Suppose it is not, and there exists x and open $V \supset \phi(x)$ such that for any open neighborhood $U \ni x$, $\phi(x_U)$ is not a subset of V for some $x_U \in U$. Choose $y_U \in \phi(x_U)$ such that $y_U \notin \phi(x)$. Clearly, x_U and y_U both are nets. The limit of x_U is x. There might be no limit for y_U, but because Y is compact there is a converging subnet with some limit y. All of y_U are in $Y \setminus V$, which is closed. Thus, y is in $Y \setminus V$ too, and $y \notin \phi(x)$. So $(x_U, y_U) \in Gr_\phi$, but a subnet of it converges to $(x, y) \notin Gr_\phi$. This means Gr_ϕ is not closed.

□

10.1. **Fixed point theorems.** This group of results is very useful, but unfortunately they mostly are very hard to prove.

Definition 10.15. If $\phi : X \rightrightarrows X$ is a correspondence, then $x \in X$ is a **fixed point** of ϕ if $x \in \phi(x)$.

Proposition 10.16. *If $\phi : X \rightrightarrows X$ has a closed graph and X is Hausdorff, then the set of all fixed points is closed.*

Proof. If x is not a fixed point, then $(x, x) \notin Gr_\phi$. There is some neighborhood $U \times U \ni (x, x)$ of (x, x) in the complement of Gr_ϕ. But then all points in U are not fixed points. □

Clearly, for functions we obtain the same old definition of a fixed point. Let us first spend some time on the case of functions.

Theorem 10.17 (Brouwer fixed-point theorem)**.** *If $f : K \to K$ is a continuous function, where $K \subset \mathbb{R}^n$ is nonempty, convex, and compact, then f has a fixed point.*

Exercise 10.18. Show that without convexity the statement is not necessarily true.

Example 10.19. For the case of $n = 1$, the statement actually is easy to prove. First, we can reduce it to the case $X = [0, 1]$. Then, consider $g : f(x) - x$. We have $g(0) \geqslant 0$ and $g(1) \leqslant 0$. By the intermediate value theorem, we know there is c such that $g(c) = 0$, but such c is a fixed point of f.

Remark 10.20. The statement can be reduced to the case of $n-$dimensional disk D^n (which is the same as the $n-$dimensional closed ball). Then, by considering the intersection of vectors $x - f(x)$ with the boundary you can express the statement in a different way: there is no continuous mapping $f : D^n \to \partial D^n$ that is constant on ∂D^n (such a mapping is called a **retraction**). Though this is quite intuitive, you would need some complicated machinery to show that such f does not exist. The logic of how you would apply this machinery is the following. If such f exists, then you can split the identical mapping as

$$id : \partial D^n \hookrightarrow D^n \xrightarrow{f} \partial D^n.$$

You thus can pass ∂D^n, the $(n-1)-$dimensional sphere, through D^n. But this turns out to be impossible, as the the disk is very primitive in some topological sense, while the sphere is not.

There are numerous generalizations to more complicated vector spaces, so if at some point of life you need a fixed point, you can open a textbook and try to find an applicable one. Now let us turn our attention to correspondences. This theorem is famous because it is used to show existence of Nash equilibria.

Theorem 10.21 (Kakutani fixed-point theorem)**.** *Suppose $\phi : X \rightrightarrows X$ is a correspondence and $X \subset \mathbb{R}^n$ is a non-empty compact convex set. If ϕ has closed graph and $\phi(x) \neq \emptyset$ and is convex for all $x \in X$, then ϕ has a fixed point.*

Though we do not prove it, it still is a good idea to draw some pictures.

Exercise 10.22. Show that without convexity or without closed graph the statement is not necessarily true.

10.2. **Selectors.** Another important thing you can do with a correspondence is selectors.

Definition 10.23. If $\phi : X \rightrightarrows Y$ is a correspondence, then function $f : X \to Y$ is a **selector** if $f(x) \in \phi(x)$ for any x.

Of course, you can always have a selector if the values of ϕ are not empty. What is more subtle is when we can have f that has good properties. Of course, in topology the most important property is continuity. For selectors, it is lower hemicontinuity that is important.

Theorem 10.24 (Michael selection theorem)**.** *Suppose $\phi : X \rightrightarrows Y$ is a lower hemicontinuous correspondence, and $\phi(x)$ is non-empty, convex, and closed for all $x \in X$. Suppose X is a metric space and Y is a complete normed vector space[8]. Then, there exists a continuous selector.*

Exercise 10.25. Show that without lower hemicontinuity the statement is not necessarily true.

[8] That is, Y is a **Banach space**.

Appendix A. Notations

It is assumed that you know some basic mathematical notation. Here is a short refreshment. $\mathbb{N}$ stands for natural numbers, $\mathbb{Z}$ stands for all integers, $\mathbb{Q}$ stands for rational numbers, and $\mathbb{R}$ stands for all real numbers. $\mathbb{R}_+$ denotes all nonnegative real numbers. If A and B are two sets, then $A \times B$ is their Cartesian product, $A \cup B$ stands for the union, and $A \cap B$ stands for the intersection. $\emptyset$ is the empty set. For a mapping f from set X to set Y we use the notation $f : X \to Y$. If you want to show how it works on particular elements of X, you use $\mapsto$. Say, for $f(x) = x^2$ this is $f : x \mapsto x^2$. Finally, if A is a subset of B, then there is a naturally arising inclusion mapping $a \mapsto a$, and it is denoted by $A \hookrightarrow B$.

Appendix B. Mathematical proofs

A proof is an argument that is convincing enough so that the person who conceived it becomes ready to convince others using the same argument. A mathematical proof is an argument for a mathematical statement, showing that a set of assumptions logically implies the conclusion.

It takes time to get used to how mathematical proofs are structured, and even more time to learn to prove things. Below are some standard methods that people use, illustrated by some famous toy examples. A proof of a non-trivial statement usually is a combination of several methods.

B.1. **Direct proof.** This is the basic one, where the conclusion is obtained by applying a sequence of logical arguments.

Example B.1. If $n \in \mathbb{Z}$ is even, then n^2 is even. If n is odd, then n^2 is odd.

Proof. If n is even, then by definition $n = 2k$ for some k. But then $n^2 = 4k^2 = 2(2k^2)$ is even too.

Similarly, if n is odd then $n = 2k + 1$ for some k. But then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd too. □

B.2. **Proof by contraposition.** With this method, one wants to prove "if A then B" but proves the contrapositive statement "if not B then not A", which is logically equivalent.

Example B.2. If $n \in \mathbb{Z}$ and n^2 is even, then n is even.

Proof. Here, A is "n^2 is even" and B is "n is even". The the contrapositive statement is "if n is odd then n^2 is odd", which you already know. □

B.3. **Proof by contradiction.** The idea is very simple. We assume that the negation of the the statement is true, do some work, and find a logical contradiction. This implies that the negation of the statement is false, and so the statement is correct.

Example B.3. $\sqrt{2} \notin \mathbb{Q}$ ($\sqrt{2}$ is not a rational number).

Proof. The negation we consider is $\sqrt{2} \in \mathbb{Q}$. Assume this is true. Then, there are natural numbers m and n with no common factor (if there was one, we could divide by it) such that

$$\sqrt{2} = \frac{n}{m}.$$

That implies $n^2 = 2m^2$. Thus, n^2 is even, and you already know that this implies that n is even and equals $2k$ for some k. Thus,

$$n^2 = 4k^2 = 2m^2 \implies m^2 = 2k^2.$$

But then m^2 is even, which implies m is even, and so both m and n have a common factor of 2. This is a contradiction. $\square$

B.4. **Proof by mathematical induction.** This is something more specific. The method can be used for statements of the type *for every natural number* $n \in \mathbb{N}$*, it holds that...* Clearly, this method cannot be applied to all theorems because often there would be no n in the statement.

The method works in two steps. First, you show that the statement is true for $n = 0$ (or $n = 1$). Then, you prove the *induction step* and show that if the statement holds for all $i \leqslant n$, it also holds for $n + 1$. Naturally, this implies that it holds for $n = 2, 3, \ldots$ and any natural n.

Example B.4. For every $n \in \mathbb{N}$,

$$1 + \ldots + n = \frac{n(n+1)}{2}.$$

Proof. For $n = 1$ this is true:

$$1 = \frac{1(1+1)}{2}.$$

Now, assume the statement is true for n. Then,

$$1 + \ldots + n + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

So the statement is correct for $n+1$ too. This was the induction step, and now the proof is complete. $\square$

B.5. **Proof by construction.** You want to show that objects with certain properties exist. To do that, you just provide an example of such an object. See the example in the next section.

B.6. **Proof by exhaustion.** There are several particular cases to consider, and you consider them one by one.

The following example will be an illustration of both a proof by construction and a proof by exhaustion.

Example B.5. There exist irrational x, y such that x^y is rational ($x, y \notin \mathbb{Q}, x^y \in \mathbb{Q}$).

Proof. Consider $\sqrt{2}^{\sqrt{2}}$. The first possible case is that $\sqrt{2}^{\sqrt{2}}$ is rational, then $x = y = \sqrt{2}$ would be a desired construction. The second possible case is that $\sqrt{2}^{\sqrt{2}}$ is irrational, then $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$ would be a desired construction:

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = 2 \in \mathbb{Q}.$$

$\square$

Appendix C. Some facts about $\mathbb{R}$

There are some basic facts about real numbers that we use several times and that allow us to say something about much more complicated spaces than $\mathbb{R}$. To add some clarity and consistency, they are listed here. We use $A \geqslant a$ to say that any element of A is larger than some number a. Similarly for other symbols.

Fact C.1 (Completeness of the real numbers)**.** *For any two nonempty sets $A, B \subset \mathbb{R}$ such that for all $a \in A$ and all $b \in B$ we have $a \leqslant b$, there is $c \in \mathbb{R}$ such that*

$$A \leqslant c \leqslant B.$$

You essentially can take it as an axiom.

Theorem C.2 (Cauchy-Cantor principle)**.** *A sequence $[a_1, b_1] \supset [a_2, b_2] \supset \dots$ of nested closed intervals has a non-empty intersection.*

Proof. For all n, m we have $a_n \leqslant b_m$. Thus, for $A = \{a_n\}_{n=1}^{\infty}$ and $B = \{a_n\}_{n=1}^{\infty}$ we can apply completeness and find c such that $A \leqslant c \leqslant B$. This means, in particular, that $a_n \leqslant c \leqslant b_n$ for all n, and thus

$$c \in \bigcap_{n=1}^{\infty} [a_n, b_n].$$

□

Exercise C.3. Why the argument is not applicable to open intervals?

Definition C.4. Consider $A \subset \mathbb{R}$. The **supremum** of A (denoted by $\sup A$) is the smallest number s such that $s \geqslant A$. The **infinium** of A (denoted by $\inf A$) is the largest number i such that $i \leqslant A$.

Completeness of the real numbers also implies the following.

Theorem C.5. *If A is bounded from above (there is some c such that $A \leqslant c$), then it has a unique supremum. If A is bounded from below (there is some c such that $A \geqslant c$), then it has a unique infinium.*

Proof. The proof is for the case of supremum. Consider B of all elements with the property. It is not empty because C is bounded. There should be some c such that $A \leqslant c \leqslant B$. It is in B by the definition of B and it is the smallest element of B. □